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INEGALITĂȚI MATEMATICE

(Extinderi și generalizări)

Tehnici și metode de demonstrație și rafinare

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CUPRINS

Capitolul 1

IDENTITĂŢI ÎNTR-UN TRIUNGHI

Aceste identităţi sunt de o mare importanţă. Ele pot fi de folos în rezolvarea unor probleme de geometrie şi trigonometrie din manualele şcolare şi pregătirii elevilor pentru performanţă matematică.

1.1. Identităţi într-un triunghi

Aplicaţii

Să se demonstreze următoarele relaţii:

1) $\sum ab = p^2 + r^2 + 4Rr$;

2) $\sum a^2 = 2\left(p^2 - r^2 - 4Rr\right)$;

3) $\sum a^3 = 2p\left(p^2 - 3r^2 - 6Rr\right)$;

4) $\sum \sin A \sin B = \dfrac{p^2 + r^2 + 4Rr}{4R^2}$;

5) $\sum \cos A \cos B = \dfrac{p^2 - 4R^2 + r^2}{4R^2}$;

6) $\sum \cos A = \dfrac{R + r}{R}$;

7) $\sum \sin A = \dfrac{p}{R}$;

8) $\sum \dfrac{\cos A}{a} = \dfrac{p^2 - 4Rr - r^2}{4Rrp} = \dfrac{\sum a^2}{8Rrp}$;

9) $\sum \cos^2 \dfrac{A - B}{2} = \dfrac{p^2 + 4R^2 + 2Rr + r^2}{4R^2}$;

10) $\sum \sin^2 \dfrac{A - B}{2} = \dfrac{8R^2 - 2Rr - r^2 - p^2}{4R^2}$;

11) $\sum \cos(A - B) = \dfrac{p^2 - 2R^2 + 2Rr + r^2}{2R^2}$;

12) $\sum \frac{m_a^2}{bc} = \frac{p^2 + 5r^2 + 2Rr}{8Rr}$;

13) $\sum \operatorname{tg} \frac{A}{2} = 2(2R - r)$;

14) $\sum \operatorname{ctg} A \operatorname{ctg} B = 1$;

15) $\sum \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{B}{2} = 1$;

16) a) $\prod \sin \frac{A}{2} = \frac{r}{4R}$;

b) $\prod \cos \frac{A}{2} = \frac{p}{4R}$;

17) $\prod \sin \frac{A-B}{2} = \frac{\prod(a-b)}{16R^2 r}$;

18) $\prod \cos \frac{A-B}{2} = \frac{p^2 + 2Rr + r^2}{8R^2}$;

19) $\sum m_a^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right)$;

20) $\sum \frac{a}{b+c-a} = 2\frac{R}{r} - 1$;

21) $\sum a \cos A = \frac{2pr}{R}$;

22) $\cos A \cos B \cos C = \frac{p^2 - 4Rr - r^2 - 4R^2}{4R^2}$;

23) $\prod(a-b)^2 = 4r^2[-p^4 + 2(10Rr + 2R^2 - r^2)p^2 - r(4R+r)^3]$;

24) $\operatorname{tg} \frac{A}{2} + \operatorname{tg} \frac{B}{2} + \operatorname{tg} \frac{C}{2} = \frac{4R+r}{p}$;

25) $\prod \operatorname{tg} \frac{A}{2} = \frac{r}{p}$;

26) $\sum \sin 2A = \frac{2pr}{R^2}$;

27) $\sum a^2 = \sum(a-b)^2 + 4S \sum \operatorname{tg} \frac{A}{2}$;

28) $\sum \operatorname{tg} A = \frac{2pr}{p^2 - 4Rr - r^2 - 4R^2}$;

29) $\sum \operatorname{ctg}\frac{A}{2}=\frac{p}{r}$;

30) $\sum \operatorname{ctg} A=\frac{p^2-r^2-4Rr}{2rp}=\frac{\sum a^2}{4rp}$;

31) $\prod \sin A=\frac{S}{2R^2}$;

32) $\sum \operatorname{tg} A \operatorname{tg} B=\frac{p^2-r^2-4Rr}{p^2-4Rr-r^2-4R^2}$;

33) $\prod \operatorname{ctg} A=\frac{p^2-4Rr-r^2-4R^2}{2pr}$;

34) $(4m_a m_b m_c)^2=p^6+(33r^2-12Rr)p^4-(33r^4+60R^2r^2-120Rr^3)p^2-$
$-(r^2+4Rr)^3$;

35) $3abc-\sum a^2(b+c-a)=4S(R-2r)$;

36) $\sum\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}=\frac{p^2+16R^2+8Rr+r^2}{4Rp}$;

37) $\sum \operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}=1-\frac{2r(4R+r)}{p^2}$;

38) $\sum \operatorname{tg}^3\frac{A}{2}\cdot\operatorname{tg}^3\frac{B}{2}=1-\frac{12Rr}{p^2}$;

39) $\sum \operatorname{tg}^4\frac{A}{2}\cdot\operatorname{tg}^4\frac{B}{2}=1+\frac{2r^2(4R+r)^2}{p^4}-\frac{16Rr}{p^2}$;

40) $\sum \operatorname{tg}^5\frac{A}{2}\cdot\operatorname{tg}^5\frac{B}{2}=1+\frac{20Rr(4R+r)^2}{p^4}-\frac{20Rr}{p^2}$;

41) $\sum \operatorname{tg}^2\frac{A}{2}=\frac{(4R+r)^2}{p^2}-2$;

42) $\sum \operatorname{tg}^3\frac{A}{2}=\frac{(4R+r)^3}{p^3}-\frac{12R}{p}$;

43) $\sum \operatorname{tg}^4\frac{A}{2}=2-\frac{16R(4R+r)}{p^2}+\frac{(4R+r)^4}{p^4}$.

Soluții

1) Considerăm identitatea:

$(x-a)(x-b)=x^3-\sum ax^2+\sum abx-abc,\ (\forall)\ x\in\mathbb{R};$

Pentru $x=p$ vom obține:

$(p-a)(p-b)(p-c)=-p^3+p\sum ab-4Rrp$, iar din formula lui Heron:

$S^2=p(p-a)(p-b)(p-c)$ rezultă: $\dfrac{s^2}{p}=-p^3+p\sum ab-4Rrp$.

Efectuând calculele, în final vom obține: $\sum ab=p^2+r^2+4Rr$.

2) $\sum a^2=\left(\sum a\right)^2-2\sum ab=4p^2-2\left(p^2+r^2+4Rr\right)=2\left(p^2-r^2-4Rr\right)$.

3) În egalitatea de la afirmația **1)**, punând $x=a$, $x=b$, $x=c$, prin însumare, obținem: $\sum a^3-2p\sum a^2+2p\sum ab-3abc=0$.

Rezultă:

$\sum a^3=2p\left(\sum a^2-\sum ab+6Rr\right)=2p\left(2p^2-2r^2-8Rr-p^2-r^2-4Rr+6Rr\right)$ sau

$\sum a^3=2p\left(p^2-3r^2-6Rr\right)$.

4) Rezultă din teorema sinusurilor și afirmația **1)**.

5) $\sum\cos(A+B)=\sum\cos A\cos B-\sum\sin A\sin B$.

Rezultă: $\sum\cos A\cos B=\sum\sin A\sin B-\sum\cos A$.

$$\text{Dar } \sum\cos A=\frac{1}{2}\sum\frac{a^2+b^2+c^2-2a^2}{bc}=\frac{a^2+b^2+c^2}{2}\sum\frac{1}{bc}-\sum\frac{a^2}{bc}=$$

$$=\frac{p\sum a^2}{4Rrp}-\frac{\sum a^3}{4Rrp}=\frac{2p\left(p^2-r^2-4Rr\right)-2p\left(p^2-3r^2-6Rr\right)}{4Rrp}=\frac{2r^2+2Rr}{2Rr}=\frac{R+r}{R}.$$

Utilizând și afirmația **4)** rezultă:

$$\sum\cos A\cos B=\frac{p^2+r^2+4Rr}{4R^2}-\frac{R}{r}=\frac{p^2+r^2-4R^2}{4R^2}.$$

7) Rezultă din teorema sinusurilor.

$$\text{8) Avem: } \sum\frac{\cos A}{a}=\sum\frac{b^2+c^2+a^2}{2abc}=\frac{1}{8pRr}\sum a^2=\frac{p^2-r^2-4Rr}{4Rr}.$$

$$\text{9) } \sum\cos^2\frac{A-B}{2}=\sum\frac{1+\cos(A-B)}{2}=\frac{3}{2}+\frac{1}{2}\left(\sum\cos A\cos B+\sum\sin A\sin B\right)=$$

$$=\frac{3}{2}+\frac{1}{2}\left(\frac{p^2+r^2+4Rr}{4R^2}+\frac{p^2+r^2+4Rr}{4R^2}\right)=\frac{12R^2+2p^2+2r^2+4Rr-4R^2}{4R^2}=$$

$$=\frac{p^2+4R^2+2Rr+r^2}{4R^2}.$$

10) $\sum \sin^2 \frac{A-B}{2} = 3 - \sum \cos^2 \frac{A-B}{2} = \frac{8r^2 - 2Rr - r^2 - p^2}{4R^2}$, unde $\sum \cos^2 \frac{A-B}{2}$ rezultă din afirmația **9)**.

11) $\sum \cos(A-B) = \sum \cos A \cos B + \sum \sin A \sin B = \frac{p^2 - 4R^2 + r^2}{4R^2} + \frac{p^2 + r^2 + 4Rr}{4R^2} =$

$= \frac{p^2 + 2R^2 + 2Rr + r^2}{2R^2}$.

12) $\sum \frac{m_a^2}{bc} = \sum \frac{2(b^2+c^2) - a^2}{4bc} = \sum \frac{b^2 + c^2 - a^2}{4bc} + \sum \frac{b^2 + c^2}{4bc} =$

$= \sum \frac{bc \cos A}{2bc} + \sum \frac{b^2 + c^2}{4bc} = \frac{1}{2} \sum \cos A + \frac{1}{4} \sum \left(\frac{b}{c} + \frac{c}{b} \right) =$

$= \frac{1}{2} \frac{R+r}{R} + \frac{1}{4} \left[(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - 3 \right] = \frac{2r - R}{4R} + \frac{p(ab+bc+ca)}{2abc} =$

$= \frac{2r-R}{4R} + \frac{p^2 + r^2 + 4Rr}{8Rr} = \frac{p^2 + 5r^2 + 2Rr}{8Rr}$.

13) $\sum \mathrm{tg} \frac{A}{2} = \sum a \frac{r}{p-a} = r \sum \frac{a}{p-a} = r \frac{\sum a(p-b)(p-c)}{pr^2} = \frac{1}{S} \{ a[p^2 - (b+c)p + bc] \} =$

$= \frac{1}{S}[2p^3 - 2p(p^2 + r^2 + 4Rr) + 12Rrp] = 2(2R - r)$.

14) $\sum \mathrm{ctg}\, A\, \mathrm{ctg}\, B = \sum \frac{1}{\mathrm{tg}\, A\, \mathrm{tg}\, B} = \frac{\sum \mathrm{tg}\, A}{\prod \mathrm{tg}\, A} = 1$.

Dar din afirmațiile **28)** și **33)** rezultă cerința.

15) $\sum \mathrm{tg} \frac{A}{2} \mathrm{tg} \frac{B}{2} = \sum \frac{r^2}{(p-b)(p-a)} = r^2 \frac{(3p - 2p)p}{S^2} = \frac{r^2 p^2}{S^2} = 1$.

16) $\prod \sin \frac{A}{2} = \frac{(p-a)(p-b)(p-c)}{abc} = \frac{r}{4R}$.

17) $\prod \sin \frac{A-B}{2} = \prod \left(\sin \frac{A}{2} \cos \frac{B}{2} - \sin \frac{B}{2} \cos \frac{A}{2} \right) =$

$= \prod \left(\frac{p-c}{c} \sqrt{\frac{p(p-c)}{ab}} - \frac{p-a}{c} \sqrt{\frac{p(p-c)}{ab}} \right) =$

$= \frac{pS}{(abc)^2} \prod (a-b) = \frac{pS}{16R^2 S^2} \prod (a-b) = \frac{1}{16R^2 r} \prod (a-b) = \frac{\prod (a-b)}{16R^2 r}$.

18) $\Pi\cos\frac{A-B}{2}=\Pi\cos\left(\frac{A}{2}-\frac{B}{2}\right)=\Pi\left(\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\right)=$

$$=\Pi\left(\frac{p}{c}\sqrt{\frac{(p-a)(p-b)}{ab}}+\frac{p-c}{c}\sqrt{\frac{(p-a)(p-b)}{ab}}\right)=\Pi\sqrt{\frac{(p-a)(p-b)}{ab}}\,\Pi\frac{2p-c}{c}=$$

$$=\frac{S^2}{p4RS}\frac{\Pi(2p-c)}{4RS}=\frac{\Pi(2p-c)}{16R^2p}=\frac{8p^3-8p^3\sum ab-abc}{16R^2p}=$$

$$=\frac{2p\left(p^2+r^2+4Rrp\right)-4Rrp}{16R^2p}=\frac{p^2+2Rr+r^2}{8R^2}.$$

20) $\sum\frac{a}{b+c-a}=\frac{1}{2}\sum\frac{a}{p-a}=\frac{1}{2}\frac{\sum a(p-b)(p-c)}{(p-a)(p-b)(p-c)}=\frac{1}{2}\frac{\sum a\left[p^2-(b+c)p+bc\right]}{\frac{S^2}{p}}=$

$$=\frac{p[2p^3-2p\sum ab+3abc]}{2S^2}=\frac{p(4RS-2rS)}{2S^2}=\frac{2p(2R-r)}{2pr}=\frac{2R-r}{r}=2\frac{R}{r}-1.$$

21) $\sum a\cos A=\frac{\sum a^2\left(b^2+c^2-a^2\right)}{2abc}=\frac{2\sum a^2b^2-\sum a^4}{2abc}=\frac{4\sum a^2b^2-\sum a^4}{2abc}=$

$$=\frac{4[(\sum ab)^2-2abc\sum a]-\left(\sum a^2\right)^2}{2abc}=\frac{\left(p^2+r^2+4Rr\right)^2-16Rrp^2-\left(p^2-r^2-4Rrp\right)^2}{2Rrp}=$$

$$=\frac{4p^2r(r+4R)-16Rrp^2}{2Rrp}=\frac{2pr}{R}.$$

22) $\sum\cos A\cos B\cos C=\frac{\Pi\left(b^2+c^2-a^2\right)}{8a^2b^2c^2}=\frac{\Pi\left(x-2a^2\right)}{8a^2b^2c^2}=$

$$=\frac{x^3-2\sum a^2x^2+\sum a^2b^2x-8a^2b^2c^2}{8a^2b^2c^2}=$$

$$=\frac{-8\left(p^2-r^2-4Rr\right)^3+8\left(p^2-r^2-4Rr\right)\left[\left(p^2-r^2-4Rr\right)^2-4abcp\right]-8a^2b^2c^2}{8a^2b^2c^2}=$$

$$=\frac{8\left(p^2-r^2-4Rr\right)\left[\left(p^2-r^2-4Rr\right)^2-\left(p^2-r^2-4Rr\right)^2-4abcp\right]-8a^2b^2c^2}{8a^2b^2c^2}=$$

$$=\frac{8\left(p^2-r^2-4Rr\right)\left[4\left(r^2+4Rr\right)p^2-16Rrp^2\right]-8a^2b^2c^2}{8a^2b^2c^2}=$$

$$=\frac{8\left(p^2-r^2-4Rr\right)4rp^2\left(r^2+4R-4R\right)-8\cdot16R^2r^2p^2}{8\cdot16R^2r^2p^2}=\frac{8\left(p^2-r^2-4Rr\right)-32R^2}{32R^2}=$$

$$=\frac{p^2-r^2-4Rr-4R^2}{32R^2}.$$

23) $\Pi(a-b)^2=4r^2[-p^4+2\left(10Rr+2R^2-r^2\right)p^2-r(4R+r)]$.

Considerăm identitatea:

$f(x)=(x-a)(x-b)(x-c)=x^3-2px+\left(p^2+r^2+4Rr\right)x-4Rrp$, $(\forall)\ x\in\mathbb{R}$.

Deoarece: $f'(x)=(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)$,

notăm: $\alpha=p^2+r^2+4Rr$, $\Delta=(a-b)^2(b-c)^2(c-a)^2=-f'(a)f'(b)f'(c)$.

Rezultă: $\Delta=-(3a^2-4ap+\alpha)(3b^2-4bp+\alpha)(3c^2-4cp+\alpha)$.

Notăm: $x_1=3a^2-4ap$; $x_2=3b^2-4ap$; $x_3=3c^2-4ap$.

Deci: $\Delta=-(\alpha+x_1)(\alpha+x_2)(\alpha+x_3)=-\alpha^3+(x_1+x_2+x_3)\alpha-$

$-(x_1x_2+x_1x_3+x_3x_1)\alpha-x_1x_2x_3+x_1+x_2+x_3=$

$=3(a^2+b^2+c^2)-4p(a+b+c)=6(p^2-r^2-4Rr)-8p^2=$

$=-2p^2-6r^2-24Rr+x_1x_2+x_2x_3+x_1x_3=\Sigma(3a^2-4ap)(3b^2-4bp)=$

$=9\Sigma a^2b^2-12p\Sigma(a^2b+ab^2)+16p^2\Sigma abc.$

Dar: $\Sigma(a^2b+ab^2)=\Sigma ab(a+b)=\Sigma ab(2p-c)=2p\Sigma ab-3abc=$

$=2p(p^2+r^2+4Rr)-12Rrp=2p(p^2+r^2+4Rr-6Rr)=2p(p^2+r^2-2Rr.$

Dar: $\Sigma a^2b^2=(\Sigma ab)^2-16Rrp$,

$\Sigma x_1x_2=9(p^2+r^2+4Rr)^2-144Rrp^2-24p^2(p^2+r^2-2Rr)+16p^2(p^2+r^2+4Rr)=$

$=9(p^2+r^2+4Rr)^2+8p^2(2p^2+2r^2+8Rr-3p^2-3r^2+6Rr-18Rr)=$

$=9(p^2+r^2+4Rr)^2+8p^2(-p^2-r^2-4Rr)=(p^2+r^2+4Rr)(p^2+9r^2+36Rr)$;

$x_1x_2x_3=-abc(4p-3a)(4p-3b)(4p-3c)=$

$=-4Rrp[64p^3-6p\cdot16p^2+36p(p^2+r^2+4Rr)-27\cdot4Rrp]=$

$=-4Rrp[-32p^3+36p(p^2+r^2+4Rr)-108Rrp]$.

Rezultă: $\Delta=-(p^2+r^2+4Rr)^3+(2p^2+6r^2+24Rr)(p^2+r^2+4Rr)^2-$

$-(p^2+9r^2+36Rr)(p^2+r^2+4Rr)^2+16Rrp^2(p^2+9r^2+9Rr)=$

$=(p^2+r^2+4Rr)^2(-p^2-r^2-4Rr+2p^2+6r^2+24Rr-p^2-9r^2-36Rr)=$

$=(-4r^2-16Rr)(p^2+r^2+4Rr)^2+16Rrp^2(p^2+9r^2+9Rr).$

Notăm $p^2 = u$. $\Delta = 4r[4Ru(u + 9r^2 + 9Rr) - (r + 4R)(u + r^2 + 4Rr)^2] =$

$= 4r[4Ru^2 + (36Rr^2 + 36R^2r)u - (r + 4R)u^2 - (r + 4R)u \cdot (2r^2 + 8Rr) -$

$- (r + 4R)(r^4 + 16R^2r^2 + 8Rr^3) = 4r[-u^2 + 2(10Rr + 2R^2 - r^2)u - r(r + 4R)^3] =$

$= 4r^2[-p^4 + 2(10Rr + 2R^2 - r^2)p^2 - r(4R + r)^3]$

24) $\sum \operatorname{tg}\frac{A}{2} = \sum \frac{r}{p-a} = r\frac{\sum(p-a)(p-b)}{\frac{S^2}{p}} = \frac{3p^2 - 4p^2 + \sum ab}{S} = \frac{-p^2 + r^2 + 4Rr + p^2}{S} =$

$= \frac{r(4R+r)}{S} = \frac{4R+r}{p}$.

25) $\prod \operatorname{tg}\frac{A}{2} = \frac{r^3}{(p-a)(p-b)(p-c)} = \frac{r^3}{\frac{S^2}{p}} = \frac{r}{p}$.

26) Din identitatea **21)** rezultă: $\sum a\cos A = \frac{2pr}{R}$ sau $\sum 2R\sin A\cos A = \frac{2pr}{R}$ echivalentă cu $\sum \sin 2A = \frac{2pr}{R^2}$.

27) Relația se mai scrie sub forma: $\sum \operatorname{tg}\frac{A}{2} = \frac{2\sum ab - \sum a^2}{4S}$. Folosind punctele **1)** și **2)** va rezulta: $\sum \operatorname{tg}\frac{A}{2} = \frac{4R+r}{p}$, adică afirmația **24)**.

28) $\sum \operatorname{tg}A = \sum \frac{\sin A}{\cos A} = \frac{1}{2R}\sum \frac{a}{\cos A} = \frac{1}{2R}\sum \frac{2abc}{b^2+c^2-a^2} = 4S\sum \frac{1}{b^2+c^2-a^2} =$

$= \frac{4S\sum(a^2+c^2-b^2)(a^2+b^2-c^2)}{\prod(b^2+c^2-a^2)} = \frac{S\sum[a^4-(b^2-c^2)^2]}{2a^2b^2c^2\cos A\cos B\cos C} =$

$= \frac{2\sum b^2c^2 - \sum a^4}{32R^2S\cos A\cos B\cos C}$. Ținând seama de **21)** și **22)**, vom obține:

$\sum \operatorname{tg}A = \frac{p^2r^2}{2R^2S\cos A\cos B\cos C} = \frac{2pr}{p^2 - 4Rr - r^2 - 4R^2}$.

29) $\sum \operatorname{ctg}\frac{A}{2} = \sum \frac{1}{\operatorname{tg}\frac{A}{2}} = \frac{\sum \operatorname{tg}\frac{B}{2}\operatorname{tg}\frac{C}{2}}{\prod \operatorname{tg}\frac{A}{2}} = \frac{1}{\frac{r}{p}} = \frac{p}{r}$.